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A NOTE ON SOME INTEGRODIFFERENTIAL INEQUALITIES

BY

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Dedicated to Professor Tsing-houa Teng on his 80th birthday

1. **Introduction.** Recently, Pachpatte [2] has applied a generalized Opial inequality [3] to obtain some integrodifferential inequalities of the Gronwall type. We will prove in this note a more general Opial inequality, and use it to derive a new integrodifferential inequality of the Gronwall type.

2. **Inequalities of the Opial type.** We first prove a general Opial inequality as following:

Theorem 1. If p is a bounded, positive and nonincreasing function on $[a, b]$, and if y is an absolutely continuous function on $[a, b]$ such that $y(a)=0$, then

$$\int_a^b p(x)|y(x)|^m|y'(x)|^n\,dx \le \frac{n(b-a)^m}{m+n}\int_a^b p(x)|y'(x)|^{m+n}\,dx,$$

where $m, n \ge 1$.

Proof. The case $m=n=1$ was proved in [1], so we may assume that $m, n>1$. Define

$$z(x)=\int_a^x (p(t))^{n/(m+n)}|y'(t)|^n\,dt, \qquad x\in[a, b].$$

Then

$$z'(x)=(p(x))^{n/(m+n)}|y'(x)|^n, \qquad x\in[a, b].$$

Using Hölder's inequality, we have

$$\begin{aligned}|y(x)| &\le \int_a^x |y'(t)|\,dt\\ &\le \left(\int_a^x (p(t))^{n/(m+n)}|y'(t)|^n\,dt\right)^{1/n}\\ &\qquad\cdot\left(\int_a^x (p(t))^{(-n)/((m+n)(n-1))}\,dt\right)^{(n-1)/n}\\ &\le (z(x))^{1/n}(p(x))^{(-1)/(m+n)}(b-a)^{(n-1)/n},\end{aligned}$$

so that

$$\begin{aligned}&\int_a^b p(x)|y(x)|^m|y'(x)|^n\,dx\\ &\quad\le \int_a^b (p(x))^{n/(m+n)}(z(x))^{m/n}(b-a)^{m(n-1)/n}|y'(x)|^n\,dx\\ &\quad= (b-a)^{m(n-1)/n}\int_a^b (z(x))^{m/n}z'(x)\,dx\\ &\quad= \frac{n(b-a)^{m(n-1)/n}}{m+n}\left(\int_a^b (p(x))^{n/(m+n)}|y'(x)|^n\,dx\right)^{(m+n)/n}.\end{aligned}$$

But, using Hölder's inequality again, we have

$$\begin{aligned}&\int_a^b (p(x))^{n/(m+n)}|y'(x)|^n\,dx\\ &\qquad\le \left(\int_a^b dx\right)^{m/(m+n)}\left(\int_a^b p(x)|y'(x)|^{m+n}\,dx\right)^{n/(m+n)}.\end{aligned}$$

Hence we have the desired inequality.

By define

$$z(x) = \int_x^b (p(t))^{n/(m+n)}|y'(t)|^n\,dt, \qquad x \in [a, b],$$

and using a similar argument as in the proof of Theorem 1, we have

Theorem 2. If p is a bounded, positive and nondecreasing function on $[a, b]$, and if y is an absolutely continuous function on $[a, b]$ such that $y(b) = 0$, then

$$\begin{aligned}&\int_a^b p(x)|y(x)|^m|y'(x)|^n\,dx\\ &\qquad\le \frac{n(b-a)^m}{m+n}\int_a^b p(x)|y'(x)|^{m+n}\,dx,\end{aligned}$$

where $m, n \ge 1$.

Combining the preceding theorems, we have

Theorem 3. If p is a bounded, positive and monotonic function on $[a, b]$, and if y is an absolutely continuous function on $[a, b]$ such that $y(a) = y(b) = 0$, then

$$\int_a^b p(x)|y(x)|^m|y'(x)|^n\,dx \leq \frac{n(b-a)^m}{m+n}\int_a^b p(x)|y'(x)|^{m+n}\,dx,$$

where $m, n \geq 1$.

We note that when $m = n = 1$. $p(x) = 1\ \forall\ x \in [a, b]$, then Theorem 3 is the original Opial inequality [1]. Also, when $p(x) = 1\ \forall\ x \in [a, b]$, then Theorem 3 becomes Theorem 6 of [3].

3. **Inequalities of the Gronwall type.** The following theorem is an application of Theorem 1:

Theorem 4. If p and y satisfy the conditions of theorem 1, if f and g are nonnegative functions such that

$$\begin{aligned}|y'(x)| \leq K &+ \int_a^x f(t)|y'(t)|\,dt \\ &+ \int_a^x g(t)\left(\int_a^t p(s)|y(s)|^m|y'(s)|^n\,ds\right)dt\end{aligned} \tag{1}$$

holds for $x \in [a, b]$, where K is a positive constant and $m, n \geq 1$, if

$$\begin{aligned}R(x) = 1 &- (m+n-1)K^{m+n-1} \\ &\cdot \int_a^x p(t)\exp\left[(m+n-1)\int_a^t r(s)\,ds\right]dt > 0\end{aligned}$$

for $x \in [a, b]$, where

$$r(x) = f(x) + \frac{n(x-a)^m}{m+n}g(x),$$

then

$$\begin{aligned}|y'(x)| \leq K &\exp\left(\int_a^x q(t)\,dt\right) \\ &+ \frac{n}{m+n}\int_a^x (t-a)^m g(t)Q(t)\exp\left(\int_t^x q(s)\,ds\right)dt,\end{aligned}$$

for $x \in [a, b]$, where

$$Q(x) = K(R(x))^{1-m-n} \cdot \exp\left(\int_a^x r(t)\,dt\right), \quad q(x) = f(x) - \frac{n(x-a)^m}{m+n} g(x).$$

Proof. For $x \in [a, b]$, let $u(x)$ be the right member of (1). Then

$$|y'(x)| \le u(x)\,, \tag{2}$$

and

$$u'(x) = f(x)|y'(x)| + g(x)\int_a^x p(t)|y(t)|^m |y'(t)|^n\,dt, \qquad u(a) = K.$$

Using Theorem 1 and (1), we have

$$u'(x) \le f(x)\,u(x) + \frac{n(x-a)^m}{m+n} g(x)\int_a^x p(t)(u(t))^{m+n}\,dt\,.$$

Adding $(n(x-a)^m/m+n)\,g(x)\,u(x)$ to both sides of the above inequality, we have

$$\begin{aligned} u'(x) + \frac{n(x-a)^m}{m+n} g(x)u(x) &\le f(x)\,u(x) \\ &+ \frac{n(x-a)^m}{m+n} g(x)\left[u(x) + \int_a^x p(t)(u(t))^{m+n}\,dt\right], \end{aligned} \tag{3}$$

which implies

$$\begin{aligned} u'(x) &\le f(x)\,u(x) \\ &+ \frac{n(x-a)^m}{m+n} g(x)\left[u(x) + \int_a^x p(t)(u(t))^{m+n}\,dt\right]. \end{aligned} \tag{4}$$

Let

$$v(x) = u(x) + \int_a^x p(t)(u(t))^{m+n}\,dt\,. \tag{5}$$

Then $v(a) = K$ and

$$u(x) \le v(x)\,. \tag{6}$$

Differentiating (5) and using (4) and (5) we see that the inequality

$$v'(x) \le r(x)\,v(x) + p(x)(v(x))^{m+n}$$

is satisfied, which is equivalent to

(7) $$(v(x))^{-m-n}v'(x)-r(x)(v(x))^{1-m-n}\le p(x).$$

Let $w(x)=(v(x))^{1-m-n}/(1-m-n)$.
Then $w(a)=K^{1-m-n}/(1-m-n)$ and (7) becomes

$$w'(x)+(m+n-1)\,r(x)\,w(x)\le p(x),$$

which implies the estimation for w such that

$$w(x)\exp\left[(m+n-1)\int_a^x r(t)\,dt\right] \le \frac{K^{1-m-n}}{1-m-n}+\int_a^x p(t)\exp\left[(m+n-1)\int_a^t r(s)\,ds\right]dt.$$

Hence

$$\begin{aligned}(v(x))^{1-m-n} &\ge \left\{K^{1-m-n}+(1-m-n)\int_a^x p(t)\cdot\exp\left[(m+n-1)\int_a^t r(s)\,ds\right]dt\right\}\\ &\qquad\cdot\exp\left[(1-m-n)\int_a^x r(t)\,dt\right]\\ &= K^{1-m-n}R(x)\exp\left[(1-m-n)\int_a^x r(t)\,dt\right],\end{aligned}$$

so that

$$v(x)\le Q(x).$$

Now using (3) we have

$$u'(x)-q(x)\,u(x)\le\frac{n(x-a)^m}{m+n}\,g(x)\,Q(x),$$

which implies the estimation for u such that

$$u(x)\exp\left(-\int_a^x q(t)\,dt\right)\le K+\frac{n}{m+n}\int_a^x (t-a)^m g(t)\,Q(t)\cdot\exp\left(-\int_a^t q(s)\,ds\right)dt.$$

Thus

(8) $$u(x)\le K\exp\left(\int_a^x q(t)\,dt\right)+\frac{n}{m+n}\int_a^x (t-a)^m g(t)\,Q(t)\cdot\exp\left(\int_t^x q(s)\,ds\right)dt.$$

Now substituting (8) in (2), we obtain the desired inequality.

We note that Theorem 2 of [2] is the special case of Theorem 4 when $p(x)=1 \ \forall \ x \in [a, b]$. Also we note that similar results can be obtained by using Theorem 2 or Theorem 3.

References

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